Sets

Define sets in two ways

- Enumeration
- Set comprehension (predicate on membership), e.g.,

  $\{n \mid n \in \mathbf{N} \wedge \exists k \bullet k \in \mathbf{N} \wedge n = 10 \times k \wedge 0 \leq n \leq 50\}$:

  the set of natural numbers between 0 and 10 that are multiples of 10

  Otherwise: $\{n : \mathbf{N} \mid \exists k : \mathbf{N} \bullet n = 10 \times k \wedge 0 \leq n \leq 50\}$

Power set: $\wp$(A) is the set of all subsets of A (including ∅ and A itself)

U is the "*universal set*," the set of discourse

A – B is the *set difference* of A and B: everything in A that's not in B

The *complement* of set A, A′, is U - A

Set Theoretical Laws

Commutative Laws

$A \cap B = B \cap A$

$A \cup B = B \cup A$

Associative Laws

$(A \cap B) \cap C = A \cap (B \cap C)$

$(A \cup B) \cup C = A \cup (B \cup C)$

Distributive Laws

$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

De Morgan's Laws

$(A \cap B)' = A' \cup B'$

$(A \cup B)' = A' \cap B'$

Identities Involving ∅

$A \cap \emptyset = \emptyset$

$A \cup \emptyset = A$

$A - \emptyset = A$

$\emptyset - A = \emptyset$

$\emptyset' = U$

Identities Involving U

$A \cap U = A$

$A \cup U = U$

$A - U = \emptyset$

$U' = \emptyset$

Laws about Set Intersection, Union, and Inclusion

$A \cap B \subseteq A$

$A \subseteq A \cup B$

Generalized Set Operations

If *S* is a set of sets, then $\cup S$ is the union of all sets in *S*, and $\cap S$ is their intersection.

A *binary relation* R on sets A and B is a set of pairs (2-tuples) all belonging to the same Cartesian product:

$R \subseteq A \times B$, sometimes written $R : A \times B$ or $R : A \leftrightarrow B$

The set of elements in A that participate in R is the *domain* of R, and the set of elements in B that participate in R is the *range* of R.

The *inverse* of R is $\{(y,x) \mid (x,y) \in R\}$

An *n*-ary relation R on sets $A_1, \ldots, A_n$ is a subset of the Cartesian product $A_1 \times \ldots \times A_n$:

$R \subseteq A_1 \times \ldots A_n$

Properties of Binary Relations

Suppose $R: A \leftrightarrow A$.

R is *reflexive* iff $(a,a) \in R$ for all $a \in A$

R is *irreflexive* iff $(a,a) \notin R$ for all $a \in A$

R is *non-reflexive* iff it's neither reflexive nor irreflexive

R is *symmetric* iff, for any $a, b \in A$, $(a,b) \in R \Rightarrow (b,a) \in R$

R is *anti-symmetric* iff, for any $a, b \in A$, $(a,b) \in R \wedge (b,a) \in R \Rightarrow a = b$

R is *non-symmetric* iff it's neither symmetric nor anti-symmetric

R is *transitive* iff, for any $a, b, c \in R$, $(a,b) \in R \wedge (b,c) \in R \Rightarrow (a,c) \in R$

R is an *equivalence relation* iff R is reflexive, symmetric, and transitive

A set *S* of subsets of arbitrary set A if a *partition* of A iff

1. $\varnothing \notin S$
2. If $S_1, S_2 \in S$, then $S_1 \cap S_2 = \varnothing$
3. $\cup S = A$

Given an equivalence relation R on set A and $a \in A$, the *equivalence class* of *a*, denoted by $[a]$, is $\{x \mid x \in A \wedge (a,x) \in R\}$

(Note: if $(a,b) \in R$, then $[a] = [b]$.)

And the *quotient* of A by R, denoted by A/R, is $\{[a] \mid a \in A\}$

A/R is a partition of A

The *composition* of $R: A \leftrightarrow B$ and $S : B \leftrightarrow C$, denoted by $R \circ S$, is defined as

$\{(a,c) \bullet a : A, c: C \mid \exists b : B \bullet (a,b) \in R \wedge (b,c) \in S\}$

Functions

Relation $f : A \leftrightarrow B$ is a function iff

$\forall x : A; y, z : B \bullet (x,y) \in f \wedge (x,z) \in f \Rightarrow y = z$

(I.e., *f* maps a domain element to a unique value.)

Where *f* is a function, we write $f : A \rightarrow B$

A is the *domain*, B is the *codomain*. It isn't assumed that every element of B is an image under *f* of an element of A. The subset of B consisting of images of elements of A under *f* is the *range* of *f*. For $A_1 \subseteq A$, we write $f(A_1)$ for the subset of B consisting of the images under *f* of the elements of A_1.

Suppose $f : A \rightarrow B$

For $C \supseteq A$, f is a *partial* function on C and is *total* iff C= A. It's *undefined* for C – A.

f is *surjective* (*onto*) if B is its range

f is *injective* (*1-to-1*) iff its inverse is a function (i.e., each element in its range has a unique pre-image)

f is *bijective* iff it's surjective and injective (then there's a 1-to-1 correspondence between A and B)

If $f : A \rightarrow B$ and $g : B \rightarrow C$, we write the composition of f and g applied to some $x \in A$ as $g(f(x))$

A *sequence* s on a set A of length n is a finite function from $\{0, \ldots, n\text{-}1\}$ to A. (Some take the domain to be $\{1, \ldots, n\}$.) Instead of $s(i)$, we write s_i.

(An array in a programming language is a sequence hence mathematically a function.)

The concatenation $s\hat{}t$ of sequence s of length n and sequence t of length m is a sequence of length $n+m$ such that

$s\hat{}t(i) = s(i)$ for $0 \leq i \leq n\text{-}1$ and

$s\hat{}t(i) = t(i\text{-}n)$ for $n \leq i \leq n+m\text{-}1$

A *bag* b on a set A is a function from a subset of A to the natural numbers. For $x \in A$, b(x) is the number of times x occurs in b. (A bag can be thought of as a set that allows duplicates.)